LAPLACE TRANSFORMS

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The Laplace transform of a function $F(t)$ defined for $t \ge 0$, often denoted by $\mathcal{L}\{F\}(s)$, is given by

$$g(s) \equiv \int_0^\infty F(t)\, e^{-st} dt \tag{1}$$

where s can in principle, be a real or complex number. Tables of Laplace transforms can be found in most textbooks (such as Saff and Snider, section 8.3) or online, such as at Wikipedia.

We can work out Laplace transforms from scratch, as this is usually just an exercise in integration. Here we give a few examples.

Example 1. Find the Laplace transform of

$$F(t) = 3\cos 2t - 8e^{-2t} \tag{2}$$

The integral is

$$g(s) = \int_0^\infty \left(3\cos 2t - 8e^{-2t}\right) e^{-st} dt \tag{3}$$

$$= \int_0^\infty \left[3\cos 2t\; e^{-st} - e^{-(s+2)t}\right] dt \tag{4}$$

The first integral can be done by integrating by parts twice, and the second is a standard integral of the exponential. Using Maple, we get

$$g(s) = -\frac{3s\mathrm{e}^{-st}\cos(2t)}{s^2+4} + \frac{6\mathrm{e}^{-st}\sin(2t)}{s^2+4} - \frac{8\mathrm{e}^{-st-2t}}{-s-2}\Bigg|_{t=0}^{\infty} \tag{5}$$

$$= \frac{3s}{s^2+4} - \frac{8}{s+2} \tag{6}$$

Example 2. Find the Laplace transform of

$$F(t) = 2 - e^{4t}\sin \pi t \tag{7}$$

We have

$$g(s) = \int_0^\infty \left(2 - e^{4t}\sin \pi t\right) e^{-st} dt \tag{8}$$

$$= -\frac{2\mathrm{e}^{-st}}{s} + \frac{\pi \mathrm{e}^{t(-s+4)}\cos(\pi t)}{(-s+4)^2+\pi^2} - \frac{(-s+4)\,\mathrm{e}^{t(-s+4)}\sin(\pi t)}{(-s+4)^2+\pi^2}\Bigg|_{t=0}^{\infty} \tag{9}$$

$$= \frac{2}{s} - \frac{\pi}{\pi^2+(s-4)^2} \tag{10}$$

Example 3. Find the Laplace transform of

$$F(t) = \begin{cases} 1 & t < 1 \\ 0 & t \geq 1 \end{cases} \tag{11}$$

Since $F(t) \neq 0$ only for $t \in [0,1]$, we have

$$g(s) = \int_0^1 e^{-st}dt \tag{12}$$

$$= \frac{1-\mathrm{e}^{-s}}{s} \tag{13}$$

Example 4. Find the Laplace transform of

$$F(t) = \begin{cases} 0 & t < 1 \\ 1 & 1 \leq t \leq 2 \\ 0 & t > 2 \end{cases} \tag{14}$$

This gives

$$g(s) = \int_1^2 e^{-st}dt \tag{15}$$

$$= \frac{\mathrm{e}^{-s}-\mathrm{e}^{-2s}}{s} \tag{16}$$

Example 5. Find the Laplace transform of

$$F(t) = \sin^2 t \tag{17}$$

Again, we can do this by integrating by parts a few times to get

$$g(s) = \int_0^\infty \sin^2 t\, e^{-st}dt \tag{18}$$

$$= -\frac{\mathrm{e}^{-st}}{2s} + \frac{s\mathrm{e}^{-st}\cos(2t)}{2(s^2+4)} - \frac{\mathrm{e}^{-st}\sin(2t)}{s^2+4}\Bigg|_{t=0}^{\infty} \tag{19}$$

$$= \frac{1}{2s} - \frac{s}{2(s^2+4)} \tag{20}$$

$$= \frac{2}{s\left(s^2+4\right)} \tag{21}$$

Example 6. Find the Laplace transform of

$$F(t) = \frac{1}{\sqrt{t}} \tag{22}$$

We have

$$g(s) = \int_0^\infty \frac{e^{-st}}{\sqrt{t}} dt \tag{23}$$

This can be solved using the substitution

$$\begin{aligned} u &= \sqrt{st} \\ du &= \frac{\sqrt{s}}{2\sqrt{t}} dt \\ t = 0 &\rightarrow u = 0 \\ t = \infty &\rightarrow u = \infty \end{aligned} \tag{24}$$

Then

$$g(s) = \frac{2}{\sqrt{s}} \int_0^\infty e^{-u^2} du \tag{25}$$

The remaining integral is a Gaussian integral with the value

$$\int_0^\infty e^{-u^2} du = \frac{1}{2}\sqrt{\pi} \tag{26}$$

so we have

$$g(s) = \sqrt{\frac{\pi}{s}} \tag{27}$$

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